The game of Nim and the Sprague Grundy Theorem

Introducing: The Game of Nim

There are three piles, or nim-heaps, of stones. Players 1 and 2 alternate taking off any non zero number of stones from a pile until there are no stones left

https://www.archimedes-lab.org/game_nim/play_nim_game.html

Why do we care about this game?

Some Definitions

Combinatorial games:

- There are two players.
- There is a finite set of positions available in the game
- Rules specify which game positions each player can move to.
- Players alternate moving.
- The game ends when a player can't make a move.
- The game eventually ends (it's not infinite).

Impartial games:

In this type of game, the set of allowable moves depends only on the position of the game and not on which of the two players is moving. Examples: Nim

Types of Nim

There are 2 versions of the game that have different winning conditions;

- **Normal Play**
- **Misere Play**

The Strategy - prerequisites

Possible positions in the game:

1. A game is in a P-position if it secures a win for the Previous player (the one who just moved)

   eg: (1,1,0) in normal play and (1,0,0) in a misere play game

2. A game is in a N-position if it secures a win for the Next player.

   eg: (1,0,0) in normal play and (1,1,0) in a misere play game

How can they be identified?

Define every position as N or P using backward induction

1. Every terminal position is a P pos
2. Every position that can reach a P pos is a N pos
3. Positions that move only to N pos are P

Nimber arithmetic

Nim-sum is an XOR sum of values (number of stones in each heap)

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Assumption: everyone knows what the bitwise XOR is (if not, here is a truth table)

A	B	C = A ^ B
0	0	0
0	1	1
1	0	1
1	1	0

The winning strategy

The winning strategy in normal play Nim is to finish every move with a Nim-sum of 0.

Explaining the strategy

- If the Nim-sum is 0 after a player’s turn, then the next move must change it.(prove using invertibility)
- It is always possible to make the nim-sum 0 on your turn if it wasn’t already 0 at the beginning of your turn.(hint: consider MSB)

Explaining the strategy

- If the Nim-sum is 0 after a player's turn, then the next move must change it.(prove using invertibility)
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If convinced of the above the proof, **think about misere play**

2 heaps	3 heaps	4 heaps
1 1 *	1 1 1 **	1 1 1 1 *
2 2	1 2 3	1 1 n n
3 3	1 4 5	1 2 4 7
4 4	1 6 7	1 2 5 6
5 5	1 8 9	1 3 4 6
6 6	2 4 6	1 3 5 7
7 7	2 5 7	2 3 4 5
8 8	3 4 7	2 3 6 7
9 9	3 5 6	2 3 8 9
n n	4 8 12	4 5 6 7
	4 9 13	4 5 8 9
	5 8 13	*n n m m*
	5 9 12	*n n n n*
* Only valid for normal play. ** Only valid for misère.		

From wikipedia

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From wikipedia

The proof is left as an exercise to the reader

- *I.N Herstein*
  from Topics in Algebra by Herstein

Bogus nim heap:

You can add or subtract coins.

There's some limitations to maintain finite-ness of the game which can be arbitrary. Note that this does not change how N and P positions are assigned

Some random stuff related to Nim

- What is the longest possible optimal game of Nim(you can only control one player)?

Also, you can control whether the game starts on an N-position or a P-position

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- 3-heap Nim as an Automaton

https://www.emis.de/journals/JIS/VOL17/Khovanova/khova6.pdf

Finally, The Sprague Grundy Theorem

Before that, some more prerequisites

Representing games as graphs

A game consists of a graph G = (X, F) where:

- X is the set of all possible game positions
- F is a function that gives for each $x \in X$ a subset of possible x's to move to, called followers. If F (x) is empty, the position x is terminal
- The start position is $x_0 \in X$. So player 1 moves first from x_0
- Players alternate moves. At position x, the player chooses from $y \in F(x)$
- The player confronted with the empty set F(x) loses

Note: the graph is finite and acyclic (such a graph is said to be progressively bounded)

Side note: This seems like it can be modelled as an automaton?(refer:https://arxiv.org/abs/1405.5942)
IGNORE IF YOU DON'T KNOW ANY AUTOMATA THEORY

One pile nim

To understand the generalisation i will first further restrict our game of nim

Also called the 21 counting game, one pile nim is a game of nim with one pile and a restriction on how many stones can be removed(at most 3 at a time)

Winning strategy?

The Sprague Grundy Function

We are not at the Sprague Grundy Theorem yet, the title is a deception

For those who know: it is just a MEX function

Henceforth, Sprague Grundy is abbreviated as SG except when required for dramatic effect

What is a MEX function?

The smallest non-negative value not found among the SG values of the followers of x.(Note: SG values are not defined yet. Also, this definition is specific to nim)

In general MEX is defined as M(S) = min({x: x does not belong to S}) (most non math definition I have)

What are SG values

Very clearly a circular explanation:

SG values are values assigned by the SG function

Now, to actually explain it. It is a recursive definition

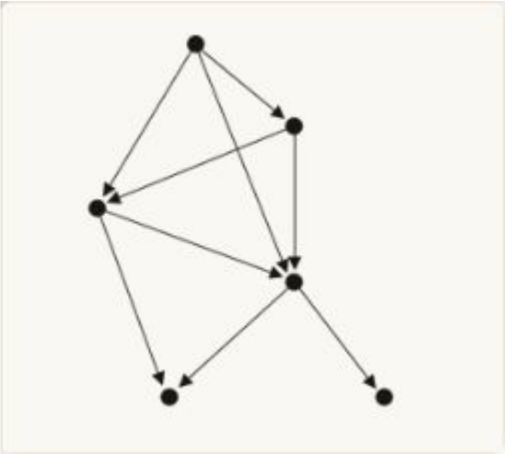
So we'll need some base cases.

Set all terminal nodes x to have g(x) = 0.

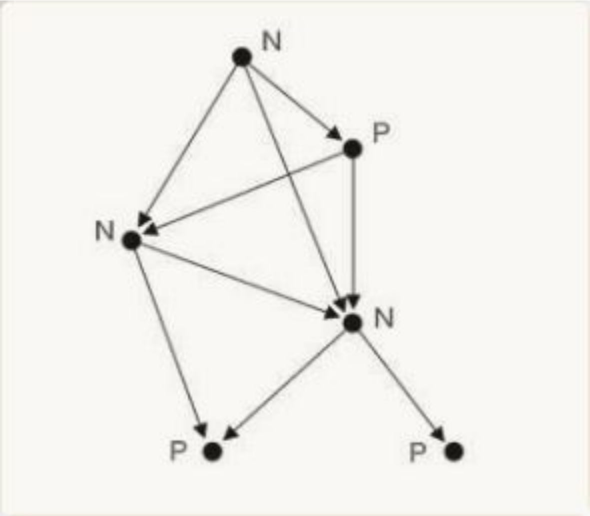
Then any nodes that have only terminal nodes as followers have g(x) = 1.

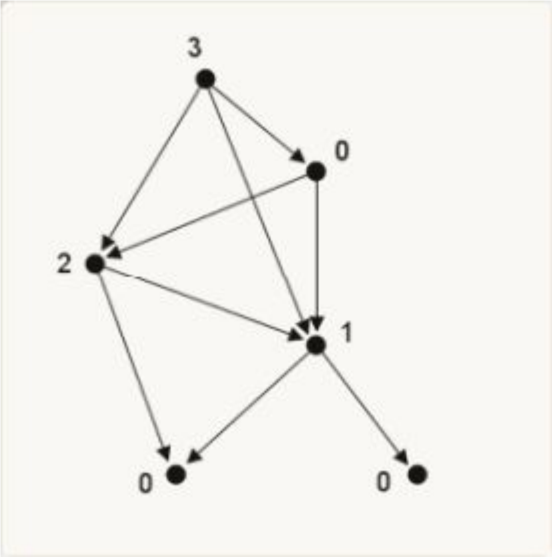
Let's try an example

On completion, label N and P positions (if I have time, I'll do it now)



In case there is no time...





Doubts?

Before we actually get to the SG Theorem

Finally, The Sprague Grundy Theorem

The Statement: Any position of an impartial game is equivalent to a nim pile of a certain size. □

Towards understanding the above very heavy statement

An equivalent statement: The SG function for a sum of games on a graph is just the Nim sum of the SG functions of its components.

Explanation:

If g_i is the Sprague-Grundy function of G_i , $i = 1 \ldots n$, then $G = G_1 + \ldots + G_n$ has Sprague-Grundy function $g(x_1 \ldots x_n) = g_1(x_1) \oplus \ldots g_n(x_n)$.

Sum of graphs(defining an algebra)

To sum the games $G1 = (X1, F1)$, $G2 = (X2, F2)$, . . . $Gn = (Xn, FN)$, $G(X, F) = G1 + G2 + \ldots + Gn$ where:

- $X = X1 \times X2 \times X3 \ldots \times Xn$, or the set of all n-tuples such that $xi \in Xi\ \forall i$
- The maximum number of moves is the sum of the maximum number of moves of each component game

The transition function is redefined to get F appropriately

Also:

$G + H = H + G$

$(G + H) + K = G + (H + K)$

The Proof (refer: https://web.mit.edu/sp.268/www/nim.pdf):

Let x = (x1 . . . xn) be an arbitrary point of X(n tuple defined as X1 x X2 x X3... where X1, X2... are the vertex sets of the component graphs). Let b = g1 (x1) ⊕ . . . ⊕ gn (xn). We are to show two things for the function g(x1 . . . xn):

1. For every non-negative integer a < b, there is a follower of (x1 . . . xn) that has g-value a.
2. No follower of (x1 . . . xn) has g-value b.

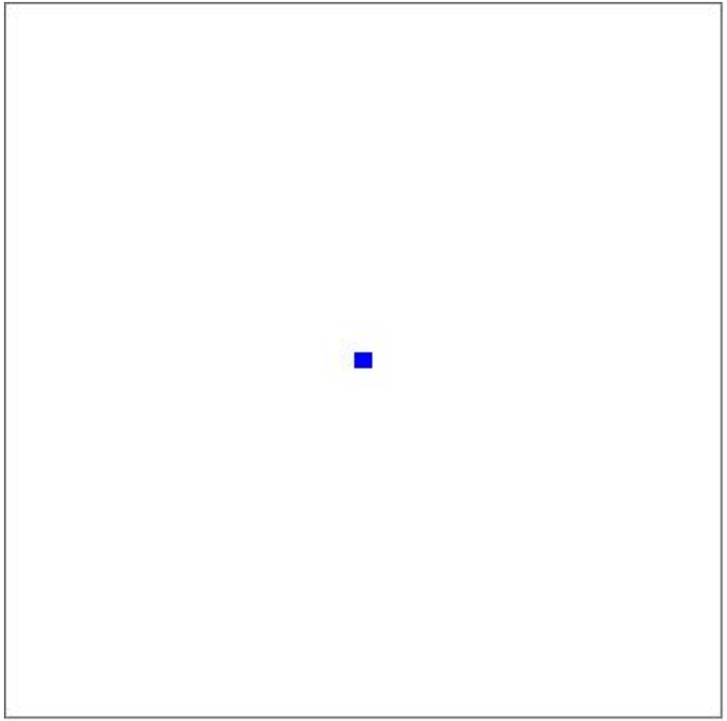
Looking at any impartial game as Nim

1. How are the two statements I gave equivalent?
2. Any game can be represented as n-heap nim?
3. Is N-heap nim equivalent to one-heap nim?

The Ulam-Warburton automaton

A 3-pile nim game can be represented as an automaton of this form

- https://www.emis.de/journals/JIS/VOL17/Khovanova/khova6.pdf



The game of Chomp

https://www.math.ucla.edu/~tom/Games/chomp.html
The above is a link to the game itself

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For a *square* starting position the strategy is obvious.

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There is no general strategy to win Chomp yet even though it has been proved that such a strategy exists.

This means we cannot assign general Grundy values to a game of Chomp, it is specific to the game

Credits

Tanya Khovanova's blog : https://blog.tanyakhovanova.com/

Lecture notes from MIT: https://web.mit.edu/sp.268/www/nim.pdf

Stuff on Chomp: https://www.win.tue.nl/~aeb/games/chomp.html

Thank You