All Alternating Sums are Secretly the Same

The Topological Background of the Euler Characteristic

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Outline

Rank Nullity Theorem

Statement

For a linear transformation $T : V \to W$,

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Noether's Isomorphism Theorems (A)

For a group homomorphism $f : G \to H$,

$$\operatorname{im} f \cong G / \ker f$$

Euler Characteristic

Graphs and Polyhedra

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General definition with Betti numbers

$$\chi = \sum_{i=0}^{\infty} (-1)^i B_i$$

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  - So, nullity$(A) = F - 1$
- We are done!

Homeomorphisms

Figure: They both have one "hole"

Simplices

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- $$C_n = \left\{ \sum_{i=0}^{n} \alpha_i u_i \;\middle|\; \sum_{i=0}^{n} \alpha_i = 1 \;\wedge\; \alpha_i \geq 0 \right\}$$

Complices?

Chain complex

Sequence of Chain groups and border maps $\{C_\bullet, \partial_\bullet\}$ such that

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \cdots$$

and

$$\partial_n \circ \partial_{n-1} = 0$$

Free Bird solo intensifies

Free Abelian group on a set S

Group $G = \langle S, + \rangle$ with a basis $B \subseteq S$ such that
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Rank of a free group

Cardinality of the basis

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- Topological results go brrrr

Homology groups

- Take a chain ocmplex and make something useful out of it
- We want a bunch of h_n where the \$n\$-th thing is the number of \$n\$-dimensional holes.
- We want some way of characterizing an \$n\$-dimensional hole using boundaries and chains.
- Enter $H_n = \ker\partial_n / \mathrm{im}\partial_{n+1}$
  - We want to classify all loops (hence $\mathrm{im}\partial_{n+1}$) based on whether they enclose a hole or not. If they don't then they should be homeomorphic to 0 (hence $\ker\partial_n$)
- Leads to another chain complex called the Homology Complex (but lite)

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- We have our Homology groups!

Graph Homologies (contd.)

- We have $\dim H_1 = \dim \ker\partial_1 - \dim \mathrm{im}\partial_2$ and $\dim H_0 = \dim \ker\partial_0 - \dim \mathrm{im}\partial_1$
- RNT also gives us $\dim C_1 = \dim \ker\partial_1 + \dim \mathrm{im}\partial_1$ and $\dim C_0 = \dim \ker\partial_0 + \dim \mathrm{im}\partial_0$
- So, we have $\dim H_1 - \dim H_0 = \dim C_1 - \dim C_0 - \dim \mathrm{im}\partial_2 + \dim \mathrm{im}\partial_0$ which is the same as $\dim H_1 - \dim H_0 = \dim C_1 - \dim C_0$
- Now, by definition, we have $\dim C_0 = V$ and $\dim C_1 = E$

Back to Grade 7 or whenever

- Now for planar graphs, we have $\dim H_1 = F - 1$ which is the number of linearly independent faces and $\dim H_0 = 1$ which is the number of connected components.
- We have our good ol' $F - E + V = 2$ (!)

Counting?? In my Algebra class??

- For planar graphs, this can also be arrived at by counting the number of edges in $E - T$ where T is a spanning tree, which gives the number of linearly independent cycles.
- This gives a nice intuition for generalizations that have nothing to do with the Euler Characteristic, but are nice to know.