A Trivial Randomized Approximation Algorithm for MAX-E3SAT

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The Problem

Given: 3SAT instance where each clause has exactly three literals.
Example:

$$(X \vee Y \vee Z) \wedge (X \vee Y \vee \overline{Z}) \wedge (X \vee \overline{Y} \vee Z) \wedge (X \vee \overline{Y} \vee \overline{Z}) \wedge$$
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Goal: Find an assignment to satisfy as many clauses as possible.

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Choose each variable to be true or false uniformly at random.

Analysis

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- $\mathbb{E}[C_i] = 1 - 1/2^{l_i}$, where l_i is the number of literals in the clause.
- Expected number of satisfied clauses is $\mathbb{E}[\sum_i C_i] = 7m/8$.

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- This would not be possible if there didn't exist at least one solution that satisfies at least $7m/8$ clauses. **Although the algorithm is randomized, our conclusion is deterministic: there always exists such a solution.**
- Non-constructive: We don't know what the solution is, but we know one always exists.

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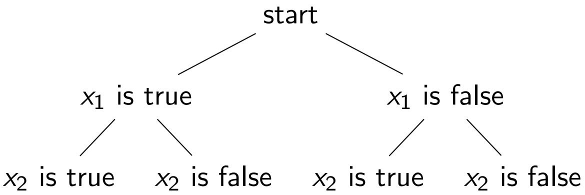

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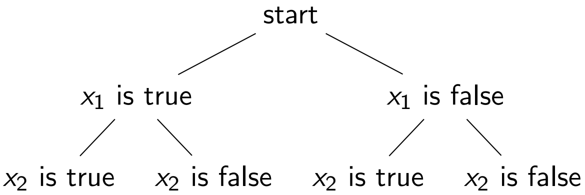

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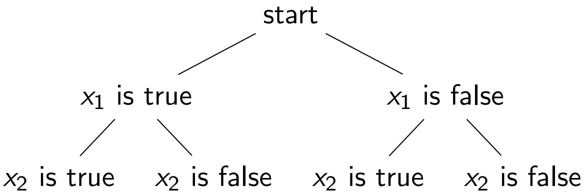

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- **Optimal:** unless $\mathrm{P} = \mathrm{NP}$, this is the best approximation ratio a polytime algorithm can achieve.

Conclusion

- Trivial randomized algorithms can do surprisingly well.
- Randomized algorithms can give deterministic conclusions.
- Some kinds of randomized algorithms can be easily derandomized.

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- $X \vee Y$ becomes $(X \vee Y \vee T) \wedge (X \vee Y \vee \overline{T})$.
- Why can't we use this to approximate 3SAT?