Two proofs in the margin

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Infinite descent

Proof by contradiction
Idea: Proposition $P(x_1, x_2, \ldots)$

Fermat's Last Theorem

$$a^n + b^n = c^n$$

1. $a, b, c \in \mathbb{Z}^+$
2. $n \in \mathbb{Z}^+, n > 2$

Fermat's Last Theorem, $n = 4$

$$a^4 + b^4 = c^4 \tag{1}$$

$$a^4 + b^4 = c^2 \tag{2}$$

Fermat's Last Theorem, $n = 4$

$a^4 + b^4 = c^2$

1. Coprime
2. Both a and b not even
3. Both a and b not odd ($\because odd^4 \equiv 1 \mod 4$)

Fermat's Last Theorem, $n = 4$

$a^4 + b^4 = c^2$

1. Coprime
2. Both a and b not even
3. Both a and b not odd ($\because odd^4 \equiv 1 \mod 4$)
4. Only possible case : one even and one odd

Fermat's Last Theorem, $n = 4$

a even, b odd

$$(a^2)^2 + (b^2)^2 = c^2$$

Thus (a^2, b^2, c) primitive Pythagorean triplet,

$$a^2 = 2mn \tag{3}$$

$$b^2 = m^2 - n^2 \tag{4}$$

$$c = m^2 + n^2 \tag{5}$$

Pythagorean triplet

$m^2 = b^2 + n^2$

b odd $\implies$ n even

$$n = 2xy \tag{6}$$

$$b = x^2 - y^2 \tag{7}$$

$$m = x^2 + y^2 \tag{8}$$

Coprimes

Equation 1 : $a^2 = 2mn$
Equation 6 : $n = 2xy$

$$m(n/2) = (a/2)^2 \tag{9}$$

Thus both m and $n/2$ squares. From 6,

$$xy = n/2 \tag{10}$$

Thus both x and y squares.
Put

$$x = u^2 \tag{11}$$

$$y = v^2 \tag{12}$$

$$m = w^2 \tag{13}$$

Descent

Equation 8 : $m = x^2 + y^2$

From 8,

$$m = x^2 + y^2 \tag{14}$$

$$\implies w^2 = u^4 + v^4 \tag{15}$$

Alternate proof

Rewriting 2,

$$a^4 = c^2 - b^4 = (c - b^2)(c + b^2) \tag{16}$$

Observations

1. No odd prime divides both
2. Both even (since *a* even and *b* odd)
3. 4 does not divide both

Therefore, $\gcd(c - b^2, c + b^2) = 2$

Alternate proof

$a^4 = (c - b^2)(c + b^2)$

Either case # 1:

$$c - b^2 = 2x^4 (x > 0 \ (\mathrm{odd})) \quad (17)$$

$$c + b^2 = 8y^4, gcd(x, y) = 1 \quad (18)$$

or case # 2:

$$c - b^2 = 8y^4 \quad (19)$$

$$c + b^2 = 2x^4, (x > 0 \ (\mathrm{odd}), gcd(x, y) = 1) \quad (20)$$

Alternate proof

Case # 1

$$c + b^2 = 8y^4, gcd(x, y) = 1 \tag{21}$$
$$c - b^2 = 2x^4 (x > 0 \ (\text{odd})) \tag{22}$$

impossible, since

$$(c + b^2) - (c - b^2) = 2b^2 = -2x^4 + 8y^4$$
$$\implies x^2 \equiv -1 \mod 4$$

Alternate proof

Descent

Case #2

$$c + b^2 = 2x^4, (x > 0 \text{ (odd)}, (x, y) = 1) \tag{23}$$

$$c - b^2 = 8y^4 \tag{24}$$

$$b^2 = x^4 - 4y^4 \implies 4y^4 = (x^2 + b)(x^2 - b)$$

By similar argument $gcd(x^2 + b, x^2 - b) = 2$. Therefore

$$x^2 - b = 2u^4 \tag{25}$$

$$x^2 + b = 2v^4 \tag{26}$$

$$\implies x^2 = u^4 + v^4 \tag{27}$$

Summary

Show that if one solution exists, smaller solution exists leading to impossible infinite chain