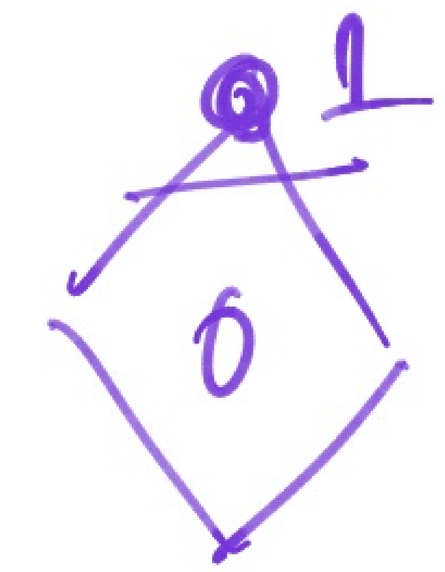
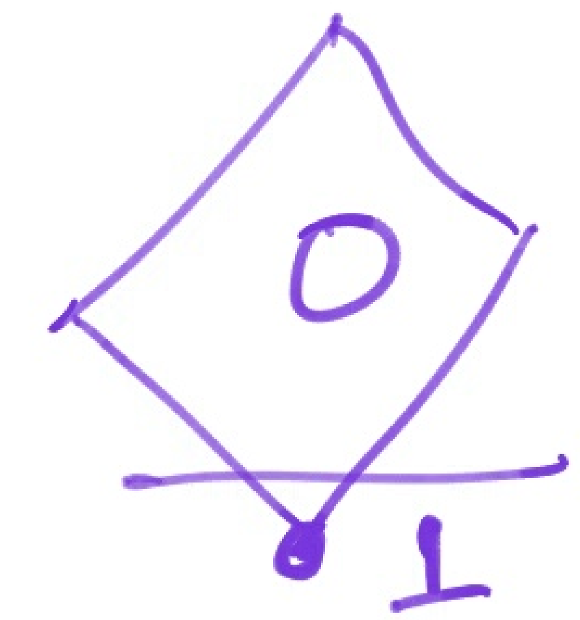
$f : \{0,1\}^n \to \{0,1\}$



polynomials in n vars. over $\mathbb{R}$

$AND : \{0,1\}^n \to \{0,1\}$

$$\prod_{i=1}^{n} x_i = AND(x_1, \ldots, x_n)$$

$h =$ 

$$g(x) = \neg f(x) = 1 - f(x)$$

$$1 - \prod_{i=1}^{n} x_i = \neg OR$$

$$h = \prod_{i=1}^{n} (1 - x_i)$$

$$OR(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$$

$\to$ Unique rep. $\to$ multilinear (deg ≤ 1 in each variable)

$f : \{0,1\}^n \to \{0,1\}$

Indicator fns (a) : $\{0,1\}^n \to \{0,1\}$

$$\prod_{i=1}^{n} \left(a_i x_i + (1-a_i)(1-x_i) \right)$$

$1\,0\,1$

$\to x_1(1-x_2)x_3$

$\underline{x_1 x_3} - x_1 x_2 x_3$

$$\boxed{f(x) = \sum_{a \in \{0,1\}^n} f(a) \cdot \underline{\text{Ind. fn.}(a)}}$$

$P_1(x) = f(x) \quad \forall x \in \{0,1\}^n$

$P_2(x) = f(x) \quad \forall x \in \{0,1\}^n$

Suppose not. $P_1(x)$, $P_2(x)$

$0 \neq r(x) = P_1(x) - P_2(x)$

$r(x) = 0 \quad \forall x$

minimal monomial in $r(x) := M$

Vars outside $M = 0$

Vars within $M = 1$

$$f(x) = \sum_{S \subseteq [n]} c_S \cdot \prod_{i \in S} x_i$$

$\leftarrow$ AND over the vars given by the set S.

$$AND_S = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}_{2^n \times 1}$$

2^n

similarly $f = \begin{pmatrix} f(x) \leftarrow x \\ \vdots \end{pmatrix}_{2^n \times 1}$

$$f : \{0,1\}^n \to \{0,1\}$$

$$x \longmapsto 1 - 2x$$

$\{0,1\} \to \{\pm 1\}$

$0 = \text{False} := 1$

$1 = \text{True} := -1$

$$f : \{\pm 1\}^n \to \{\pm 1\} \quad \left.\begin{matrix} 1 \to 0 \\ -1 \to 1 \end{matrix}\right\} \; y \to \frac{1-y}{2}$$

$$AND_S = \prod_{i \in S} x_i$$

0/1	−1/+1	
x_1	$\frac{1-x_1}{2}$	$1 - 2 \cdot \left(\frac{1-y_1}{2}\right) = y_1$

$$\text{AND}(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3 \leftarrow \{0,1\}\text{-AND}$$

$(-1,+1)$:

$$1 - 2\left(\frac{1-y_1}{2}\right)\left(\frac{1-y_2}{2}\right)\left(\frac{1-y_3}{2}\right)$$

$$= 1 - \frac{1}{4}\left(1 - y_1 - y_2 - y_3 + y_1y_2 + y_1y_3 + y_2y_3 - y_1y_2y_3\right)$$

$$= \frac{3}{4} + \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 - \frac{1}{4}y_1y_2 - \frac{1}{4}y_1y_3 - \frac{1}{4}y_2y_3 + \frac{1}{4}y_1y_2y_3$$

$$\text{AND} : \{\pm 1\}^3 \to \{\pm 1\}$$

$$\left\{ y_S = \prod_{i \in S} y_i \;\middle|\; \forall S \subseteq [n] \right\}$$

$$\uparrow : \{\pm 1\}^n \to \{\pm 1\}$$

$$\underline{y_1 \cdot y_2 \cdot y_3} = \text{Parity} \leftarrow \{\pm 1\}$$

$$\langle \chi_S, \chi_T \rangle = E_x[\chi_S(x) \cdot \chi_T(x)] = E_{x_1, \dots x_n}\left[\prod_{i \in S} x_i \cdot \prod_{i \in T} x_i\right]$$

$\underline{n = 3}$, $S \subseteq \{1, 2, 3\}$

$S = \{1\ 3\}$

$$\chi_S(x_1, x_2, x_3) = \prod_{i \in S} x_i = x_1 \cdot x_3$$

$1\ 1\ 1 \Rightarrow 1$

$1\ {-1}\ 1 \rightarrow 1$

$-1\ 1\ 1 \rightarrow -1$

$-1\ {-1}\ 1 \rightarrow -1$

$1\ 1\ {-1} \rightarrow -1$

$1\ {-1}\ {-1} \rightarrow -1$

$-1\ 1\ {-1} \rightarrow 1$

Uniform dist =

$$P(x_1 \dots x_n = a_1 \dots a_n) = \frac{1}{2^n}$$

$$= \frac{1}{2^n} \sum_x \left(\prod_{i \in S} x_i\right) \cdot \left(\prod_{i \in T} x_i\right)$$

$\underline{n = 1}$:-

$$Pr(x_1 = 1) = \frac{1}{2}$$

$$Pr(x_1 = -1) = \frac{1}{2}$$

$$E[x_1] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$$

$$E_{x_1, x_2}[x_1 x_2] = E_{x_1}[x_1] \cdot E_{x_2}[x_2]$$

$$= 0 \cdot 0 = 0$$

$$\langle \chi_S, \chi_T \rangle = \mathbb{E}_{x_1,\dots,x_n}\left[\prod_{i \in S} x_i \cdot \prod_{j \in T} x_j\right]$$

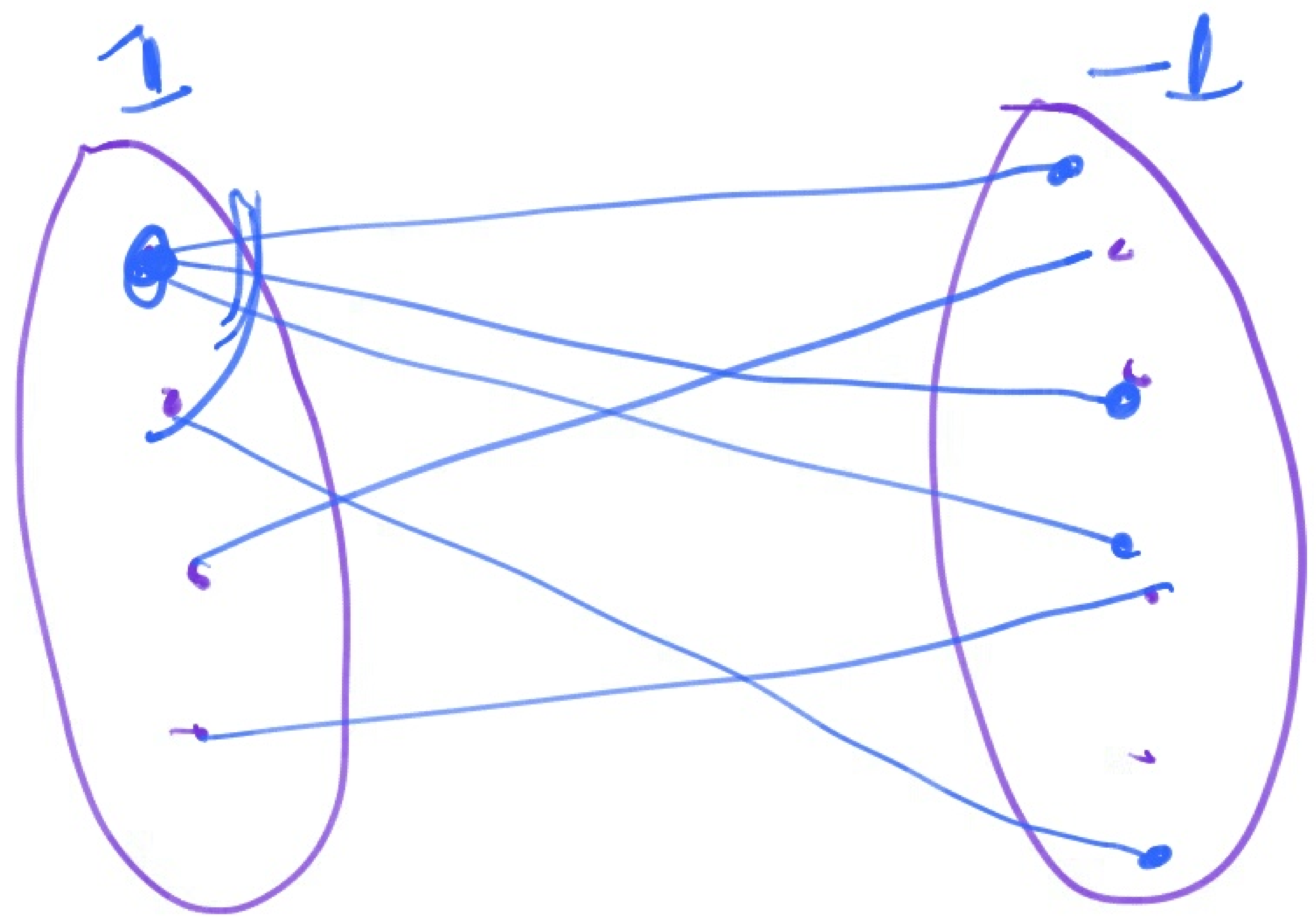
$S \neq T$

$$= \mathbb{E}_{x_1,\dots,x_n}\left[\prod_{\underbrace{i \in S \Delta T}_{\text{symmetric diff.}}} x_i \cdot \prod_{j \in S \cap T} x_j^2\right]$$

$$= \mathbb{E}_{x_1,\dots,x_n}\left[\prod_{i \in S \Delta T \neq \emptyset} x_i\right]$$

$$= \prod_{i \in S \Delta T} \mathbb{E}[x_i] = 0$$

$f : \{\pm 1\}^n \to \{\pm 1\}$



$x : f(x) = 1$

$\underline{f^{-1}(1)}$

$x : f(x) = -1$

$\underline{f^{-1}(-1)}$

avg. sens. of AND

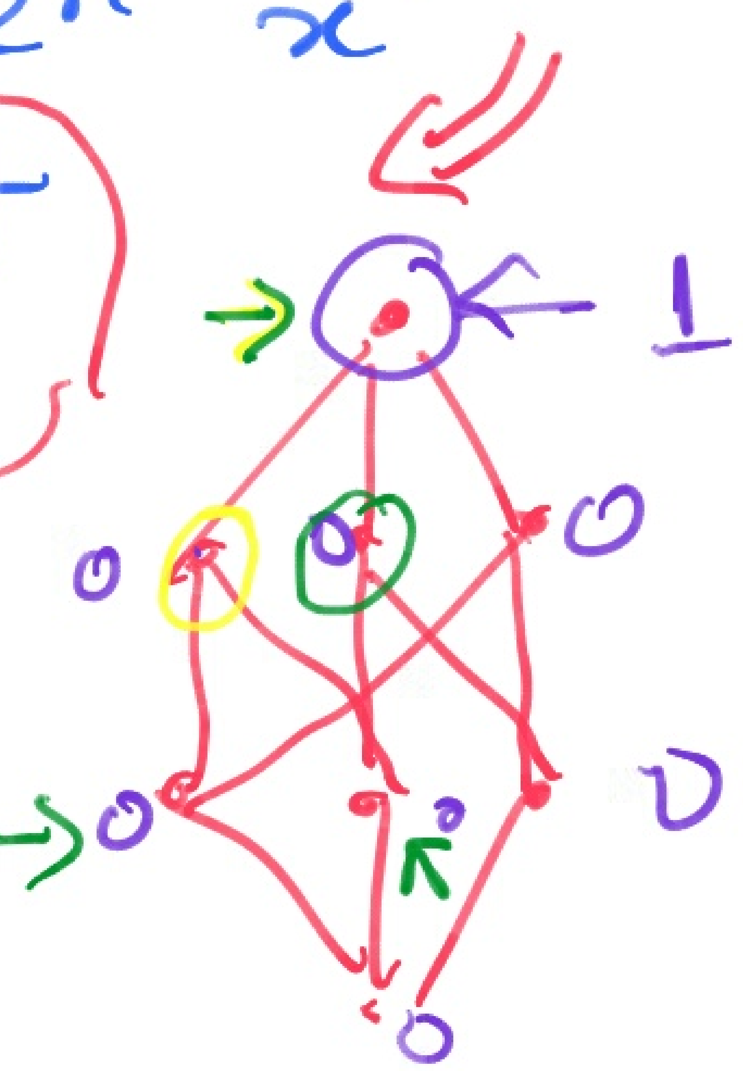
$= \frac{1}{2^n}[n + 1 \cdot n]$

$= \frac{2n}{2^n}$

$\deg(x) = S(f, x)$

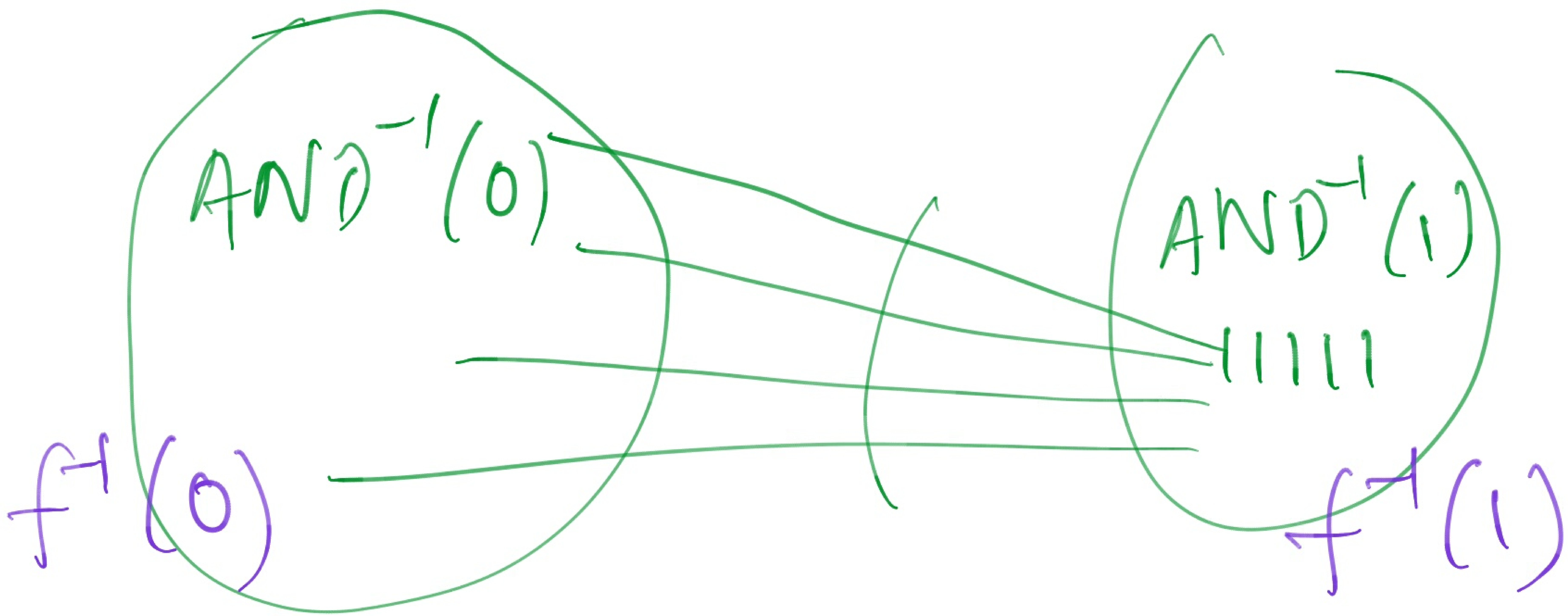
avg. sens. $= E_x[S(f, x)]$

$= \frac{1}{2^n} \sum_x S(f, x)$



Claim :- avg. sens $= \frac{2 \cdot \#\text{edges in the cut}}{2^n}$

Examples :- avg. sens (Parity) $= n$



$$\frac{2 \cdot n}{2^n}$$

$\text{Sens}(f, x) = \#$ neighbors having diff. function value

avg. sens $= \frac{1}{2^n} \sum_{x} s(f, x)$.

$f : \{0,1\}^n \to \{0,1\}$

$$\text{Inf}_i(f) = \Pr_x \left[f(x) \neq f(x^i) \right] \quad \forall\ i \in [n]$$

$$\text{Inf}(f) = \sum_{i=1}^{n} \text{Inf}_i(f) \ ;$$